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EFFECTIVE ELASTIC MODULI OF INHOMOGENEOUS MEDIA IN THE CASE OF POTENTIAL AND BIVORTICAL TENSOR FIELDS

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There is established the equivalence of methods of computing the effective elastic moduli for potential and bivortical tensor fields. It is shown that in these cases the exact solutions agree and the agreement of the solutions found in a singular approximation is possible only under definite constraints imposed on the parameters of the comparison body.

1. It is known [1] that any symmetric tensor of the second rank can be decomposed into potential and bivortical components. The potential component of elastic fields is found from the equations

$$\operatorname{div}\sigma = -f, \quad \operatorname{Rot}\varepsilon = 0, \quad \sigma = \lambda\varepsilon \quad (\operatorname{Rot} = \operatorname{rot}\oplus\operatorname{rot}) \tag{1.1}$$

where σ, ε are the stress and strain tensors, λ is the tensor of the elastic moduli, f is the vector of the volume density of the external forces. (Here and throughout wherever possible, the tensor indices are omitted). In turn, the bivortical component is determined by using the equations

$$\operatorname{Rot}\varepsilon = -\eta, \quad \operatorname{div}\sigma = 0, \quad \sigma = \lambda\varepsilon \tag{1.2}$$

where η is the tensor of incompatibility which describes the distribution of the internal stress sources.

Equations (1.1) and (1.2) are used below to compute the effective elastic moduli and the elastic fields of inhomogeneous media for which the material characteristics (the density, elastic moduli, compliances, etc.) are random fields, and f, η are regular functions of the coordinates.

Let us first examine potential fields. Together with the field $\lambda(\mathbf{r})$ we introduce the homogeneous comparison field λ_c. The displacement fields u and u_c corresponding to the tensors λ and λ_c satisfy the equations

$$Lu = -f, \quad L = \operatorname{div}\lambda\operatorname{def}, \quad u|_S = u_0 \tag{1.3}$$

$$L_c u_c = -f, \quad L_c = \operatorname{div}\lambda_c\operatorname{def}, \quad u_c|_S = u_0 \tag{1.4}$$

Here S is the surface bounding the medium, u_0 is the value of the displacement on the boundary, and because of potentiality the strains ε, ε_c are connected with the displacements u, u_c by the relations $\varepsilon = \operatorname{def} u$, $\varepsilon_c = \operatorname{def} u_c$. A medium for which the solution of (1.4) is known is selected as the comparison medium.

The problem is to find the strain ε and the tensor of the effective elastic moduli λ_* defining the mean strain $\langle\varepsilon\rangle = \operatorname{def}\langle u\rangle$ by the equation

$$L_*\langle u\rangle = -f, \quad L_* = \operatorname{div}\lambda_*\operatorname{def}, \quad \langle u\rangle|_S = u_0 \tag{1.5}$$

where the angular brackets denote the statistical average.

It can be shown [2] the the equalities

$$\langle\varepsilon\rangle = \varepsilon_c = \operatorname{def} u_0, \quad f = 0 \tag{1.6}$$

hold in the absence of external forces, and the average taken over the volume agrees with the statistical average.

Denoting the excess fields relative to the comparison body and the operators by primes, we have from (1.3) and (1.4)

$$L_c u' = -L'u, \quad u'|_S = 0 \quad (L' = L - L_c, \quad u' = u - u_c) \tag{1.7}$$

Let us introduce the Green's tensor of the operator L_c which satisfies the equation

$$L_c G = -I\delta, \quad G|_S = 0$$

where I is the unit tensor of the second rank, and δ is the delta function.

We then have from (1.7)

$$u = u_c + G * L'u \tag{1.8}$$

where the asterisk denotes the integral convolution operation. The integral equation (1.8) permits expressing the field u in terms of the known field u_c. Letting the operator def act on both sides of (1.8), we obtain an equation for the strain

$$\varepsilon = \varepsilon_c + Q\lambda'\varepsilon, \quad Q = \operatorname{def} G * \operatorname{div} \tag{1.9}$$

Solving (1.9) for ε, we find

$$\varepsilon = a\varepsilon_c, \quad a = (I - Q\lambda')^{-1} \tag{1.10}$$

In general, the operator a has the meaning of an infinite series

$$a = \sum_0^\infty y^n, \quad y = Q\lambda'$$

which is obtained in solving (1.9) by the method of iteration.

We now eliminate the field ε_c from (1.10). To do this, we take its average and express ε_c in terms of $\langle\varepsilon\rangle$

$$\varepsilon_c = \langle a\rangle^{-1}\langle\varepsilon\rangle \tag{1.11}$$

Substituting (1.11) into (1.10) yields

$$\varepsilon = A\langle\varepsilon\rangle, \quad A = a\langle a\rangle^{-1} \tag{1.12}$$

The operator A also has a representation in the form of a series

$$A = \sum_0^\infty Y^n, \quad Y = ty, \quad tF \equiv F - \langle F\rangle \tag{1.13}$$

The solution (1.12) permits finding the tensor of the effective elastic moduli in the form

$$\lambda_* = \langle\lambda A\rangle, \quad \langle A\rangle = I, \quad \langle\sigma\rangle = \lambda_*\langle\varepsilon\rangle \tag{1.14}$$

The expressions (1.12) and (1.14) therefore solve the problem about describing an inhomogeneous deformable medium completely. However, the exact solutions found in the form of operator series are not of formal nature only in rare cases. This is related to the fact that available information about the statistical properties of an arbitrary inhomogeneous medium is insufficient for the computation of λ_* and ε according to (1.12) and (1.14). Moreover, the mathematical difficulties of such a computation are still quite great. In this connection it is often necessary to limit oneself to the consideration of approximate solutions [3 — 7]. One of them is the singular approximation (the S-approximation) [5 — 7] based on extracting the singular component of the Green's tensor G, and therefore the operator Q as well. In the general case, the Green's tensor G can be represented as the difference between two terms $G = G_\infty - G_1$, one of which G_∞ corresponds to an unbounded medium, while the other takes account of the presence of the surface bounding the medium under consideration. Singularities of δ-function type originated in taking the second derivative of the tensor G. Introducing the notation [6, 7]

$$(\operatorname{def} \operatorname{div}^T)^s G_\infty = g\delta, \quad \operatorname{div}^T G * \equiv G * \operatorname{div} \qquad (1.15)$$
$$(\operatorname{def} \operatorname{div}^T)^f G * F = hF$$

where s and f denote the singular and formal parts, and T the transpose, we rewrite (1.12) in the form

$$A = (I - g\lambda')^{-1} R \langle (I - g\lambda')^{-1} R\rangle^{-1}$$
$$R = (I - h\lambda')^{-1} \langle (I - h\lambda')^{-1}\rangle^{-1} = \sum_0^\infty (th\lambda')^n$$

Here g is an ordinary tensor whose properties are determined by the structure of an inhomogeneous medium, I is the unit symmetric tensor of the fourth rank, and the operator R describes the nonlocal part of the interaction.

Since only local interactions are taken into account in the S-approximation, only the first member should be retained in the operator R. Consequently, the solution of the problem has the following form in place of (1.12) and (1.14)

$$\varepsilon = A_s \langle \varepsilon \rangle, \quad \lambda_* = \langle \lambda A_s \rangle, \quad A_s = (I - g\lambda')^{-1} \langle (I - g\lambda')^{-1}\rangle^{-1} \qquad (1.16)$$

Introducing the auxiliary tensor b_c by means of the equation

$$g(\lambda_c + b_c) = -I \qquad (1.17)$$

we obtain the following equations for A_S and λ_*:

$$A_s = (\lambda + b_c)^{-1}(\lambda_* + b_c), \quad \lambda_* + b_c = \langle (\lambda + b_c)^{-1}\rangle^{-1} \qquad (1.18)$$

The expressions (1.18) depend on the tensor λ_c, which is governing for both g and b_c. Depending on the selection of the comparison field λ_c, the upper $\lambda+$ and lower $\lambda-$ boundaries for λ_* [7] can hence be obtained.

2. Let us turn to a computation of the fields and the effective elastic moduli in the case of bivortical fields. The stress fields σ and σ_c corresponding to the tensors $s = \lambda^{-1}$ and $s_c = \lambda_c^{-1}$ satisfy the equations

$$L\sigma = -\eta, \quad \operatorname{div} \sigma = 0, \quad L = \operatorname{Rot} s, \quad \sigma_n \mid_S = \sigma_n^\circ \qquad (2.1)$$
$$L_c\sigma_c = -\eta, \quad \operatorname{div} \sigma_c = 0, \quad L_c = \operatorname{Rot} s_c, \quad \sigma_n^c \mid_S = \sigma_n^\circ \qquad (2.2)$$

where σ_n is the stress vector with components $\sigma_{kl} n_k$, and n_l is the unit vector of the external normal to the surface S bounding the medium. The tensor s_* defines the mean stress by using the equation

$$L_* \langle \sigma \rangle = -\eta, \quad \operatorname{div} \langle \sigma \rangle = 0, \quad L_* = \operatorname{Rot} s_*, \quad \langle \sigma_n \rangle \mid_S = \sigma_n^\circ \qquad (2.3)$$

As for (1.6), we have for the case $\eta = 0$

$$\langle \sigma \rangle = \sigma_c = \sigma_0, \qquad \eta = 0$$

Let us write the solution (2.1) in the form (1.9). For this, we introduce the Green's tensor Z of the operator L_c from (2.2), which satisfies the equation [1]

$$L_cZ = -\theta^0, \quad \operatorname{div} Z = 0, \quad \theta^0 = -\operatorname{Rot}\operatorname{Rot}\zeta, \quad Z_n \mid s = 0$$
$$\Delta^2\zeta = -\delta, \quad \zeta \mid s = 0$$

where θ^0 is the unit operator in the subspace of bivortical tensors, Δ is the Laplace operator, and the tensor components Z_n equal $Z_{lijk}n_l$. Then, as for (1.7) we obtain an equation from (2.1) and (2.2)

$$L_c\sigma' = -L'\sigma, \quad \operatorname{div}\sigma' = 0, \quad \sigma_n' \mid s = 0$$

which yields

$$\sigma = \sigma_c + Ps'\sigma, \quad P = Z * \operatorname{Rot} \tag{2.4}$$

Solving (2.4) for σ, we find

$$\sigma = b\sigma_c, \quad b = (I - Ps')^{-1} = \sum_0^\infty (Ps')^n$$

Hence, after eliminating the field σ_c, we find the connection between σ and $\langle\sigma\rangle$ in the form

$$\sigma = B\langle\sigma\rangle, \quad B = b\langle b\rangle^{-1} = \sum_0^\infty (tPs')^n \tag{2.5}$$

The solution (2.5) permits finding the tensor of the effective compliances s_* in the form

$$s_* = \langle sB\rangle, \quad \langle B\rangle = I, \quad \langle\varepsilon\rangle = s_*\langle\sigma\rangle \tag{2.6}$$

In the S-approximation, the field σ and the effective compliances s_* satisfy the relationships

$$\sigma = B_s\langle\sigma\rangle, \quad s_* = \langle sB_s\rangle, \quad B_s = (I - zs')^{-1}\langle(I - zs')^{-1}\rangle^{-1} \tag{2.7}$$

where the tensor z is defined as follows:

$$(\operatorname{Rot}^T)^s Z_\infty = z\theta^0, \quad Z = Z_\infty - Z_1 \tag{2.8}$$

The tensors B_s and s_* can be simplified by using an auxiliary tensor a_c introduced according to the equation

$$z(s_c + a_c) = -I \tag{2.9}$$

Using (2.9), we obtain equations for B_s and s_* from (2.8)

$$B_s = (s + a_c)^{-1}(s_* + a_c), \quad s_* + a_* = \langle(s + a_c)^{-1}\rangle^{-1} \tag{2.10}$$

Exactly as in the case of potential fields, the solution of the problem for the bivortical fields in the S-approximation will depend on the parameter s_c of the comparison body. We obtain the upper s^+ and lower s^- boundary of the field s_* [7] from (2.10) depending on the choice of the comparison field s_c.

3. Let us show that the solutions (1.12) and (1.4) for the potential fields are equivalent to the solutions (2.5) and (2.6) for the bivortical fields.

We first prove that the Green's tensor Z which has the form [1]

$$Z = \lambda_c[I + \operatorname{def} G * \operatorname{div}\lambda_c]\operatorname{Rot}\zeta * \tag{3.1}$$

satisfies the relationship

$$P = -\lambda_c - \lambda_c Q\lambda_c \tag{3.2}$$

where the definitions (1.9) and (2.4) have been taken into account. Indeed, the equation

$$Z * \operatorname{Rot} = \lambda_c[I + \operatorname{def} G * \operatorname{div}\lambda_c]\operatorname{Rot}\operatorname{Rot}\zeta *$$

together with the identity [1]

$$\Delta^2 = \text{Rot Rot} + \text{def } \pi, \quad \pi = (2\Delta - \text{grad div}) \text{ div}$$

and the equation

$$G * \text{div } \lambda_c \text{ def} = -I$$

obtained from (1. 4), yields

$$P = \lambda_c \text{ Rot Rot } \zeta * + \lambda_c \text{ def } G * \text{div } \lambda_c (\Delta^2 - \text{def } \pi) \zeta * = \lambda_c \text{ Rot Rot } \zeta * + \lambda_c \text{ def } G * \text{div } \lambda_c \Delta^2 \zeta * + \lambda_c (\Delta^2) - \text{Rot Rot}) \zeta * = \lambda_c (I + Q\lambda_c) \Delta^2 \zeta *$$

Hence, (3. 2) obviously results (because of the equation $\Delta^2\zeta = -\delta$) . Using (3.2), we find for the operator b in (2. 5)

$$b^{-1} = I - Ps' = \lambda_c s + \lambda_c Q \lambda_c s'$$

Hence, after substituting $\lambda_c s' = -\lambda' s$ and taking account of (1. 10) we have

$$b^{-1} = \lambda_c (I - Q\lambda') s, \quad b = \lambda a s_c \tag{3. 3}$$

The relationship (3. 3) interrelating the operators a and b also permits establishment of a connection between the operators A and B as well. Substituting (3. 3) into (2. 5), we obtain

$$B = \lambda A \lambda_*^{-1} \tag{3. 4}$$

Here the definitions (1. 12) and (1. 14) of the operator A and the tensor λ_* have been taken into account.

We now find the connection between λ_*, obtained by using (1. 14), and s_*, computed according to (2. 6). For this, we substitute (3. 4) into (2. 6), which yields (since $\langle A \rangle = I$)

$$s_* = \langle sB \rangle = \langle A \lambda_*^{-1} \rangle = \lambda_*^{-1} \tag{3. 5}$$

Therefore, the tensors of the effective elastic moduli agree in both cases and are independent of the nature of the fields being considered. According to (2. 5), (3. 4) and (3.5), we write for the strain field ε

$$\varepsilon = sB\langle \sigma \rangle = sB\lambda_* \langle \varepsilon \rangle = A \langle \varepsilon \rangle \tag{3. 6}$$

which agrees with (1. 12). However, despite the fact that the operator A in (1. 12) and (3. 6) is the same, the strains ε computed by means of these formulas are distinct. This is explained by the fact that the strain tensor $\langle \varepsilon \rangle$ in (1. 12) and (3. 6) satisfies different equations. In the first case it is found according to (1. 5) and in the second by using (2.3) and (2. 6). The solutions (1. 12) and (3. 6) agree only in the case $\varphi = 0, \ \eta = 0.$

4. Let us consider the solutions (1. 17) and (2. 10) obtained in the S-approximation for the potential and bivortical fields.

Only the tensors g and z, defined in terms of the Green's tensors G_∞ and Z_∞ for an inhomogeneous medium according to (1. 15) and (2. 8) are used in the S-approximation, hence it is expedient first to establish a connection analogous to (3. 2) between them. It can be shown that all the calculations in Sect. 3, associated with the derivation of (3.2), remain valid also in the examination of the operators Q_∞ and P_∞ in ζ is understood to be a Green's function of the operator Δ^2 for an unbounded medium which has the form $\zeta = r / (8\pi)$.

Performing the replacements $P \rightarrow P_\infty$ and $Q \rightarrow Q_\infty$ in (3. 2) and taking the singular component from both sides of the equality, we write

$$P_\infty^s = -\lambda_c - \lambda_c Q_\infty^s \lambda_c \tag{4. 1}$$

But the operators Q_∞^s and P_∞^s degenerate, according to (1.15) and (2.8), into the tensors g and z, respectively. Taking account of the definitions (1.17) and (2.9) and expressing Q_∞^s and P_∞^s in terms of the tensors b_c and a_c

$$Q_\infty^s = -(\lambda_c + b_c)^{-1}, \quad P_\infty^s = -(s_c + a_c)^{-1}$$

we obtain from (4.1) after simple manipulations

$$(s_c + a_c)^{-1} = (\lambda_c^{-1} + b_c^{-1})^{-1}$$

which reduces to the relationship

$$a_c b_c = I, \quad \lambda_c s_c = I \tag{4.2}$$

because of the equality $\lambda_c s_c = I$.

Now, let us transform the expression for the tensor s_* from (2.10) found in the S-approximation. Using (4.2) and the equality $\lambda s = I$, we obtain from (2.10) and (1.18)

$$s_* + a_c = \langle(\lambda^{-1} + b_c^{-1})^{-1}\rangle^{-1} = a_c\langle I - b_c(\lambda + b_c)^{-1}\rangle^{-1} = \tag{4.3}$$
$$a_c[I - b_c(\lambda_* + b_c)^{-1}]^{-1} = \lambda_*^{-1} + a_c$$

Therefore, the relationship (3.5) is also satisfied in the S-approximation, however, under the additional condition (4.2).

Let us transform the tensor B_s. It is seen that by taking account of (4.2) and (4.3), the tensors B_s from (2.10) and A_s from (1.18) satisfy the equality

$$B_s = \lambda A_s \lambda_*^{-1} \tag{4.4}$$

Substituting (4.4) into (2.7), we find for the strains

$$\varepsilon = sB_s\langle\sigma\rangle = A_s\lambda_*^{-1}\lambda_*\langle\varepsilon\rangle = A_s\langle\varepsilon\rangle \tag{4.5}$$

This establishes the equivalence of computing the potential (1.16) and the bivortical (4.5) fields in the S-approximation. However, as in the case of the exact solutions (see Sect. 3), this does not mean equality of the field (1.16) to field (4.5).

In conclusion, let us note that using the mean values $\langle\lambda\rangle$ and $\langle s\rangle$ as λ_c and s_c does not satisfy the relationship (4.2), and therefore, does not lead to (4.3) — (4.5).

Both computation schemes lead to identical results for both the tensor of the effective elastic moduli and for the fields in the solution of (1.1) and (1.2) without right sides, i.e., for $f = 0$ and $\eta = 0$.

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